Theorem. *Let b and m be integers greater than 1. If $\frac{1}{m} = (0.a_1a_2\cdots a_i\cdots)_b$, then for any $t \in \mathbb{N}$, in base $(b+mt)$, the fraction $\frac{1}{m}$ has the digital representation*

$$\tfrac{1}{m} = (0.a_1'a_2'\cdots a_i'\cdots)_{b+mt},$$

where $a_i' = a_i + tk_i$ with $k_i = (b^{i-1} \pmod{m})$.

Proof. By the lemma,

$$a_i = \frac{b}{m}(b^{i-1} \pmod{m}) - \frac{1}{m}(b^i \pmod{m}), \text{ and}$$
$$a_i' = \frac{b+mt}{m}((b+mt)^{i-1} \pmod{m}) - \frac{1}{m}((b+mt)^i \pmod{m}).$$

On the other hand, $(b+mt)^{i-1} \equiv b^{i-1} \pmod{m}$ and $(b+mt)^i \equiv b^i \pmod{m}$. Thus we get $a_i' - a_i = t(b^{i-1} \pmod{m}) = tk_i$, as claimed. ■

Earlier we discussed $\frac{1}{7}$ in bases 3, 10, and 17. As a second example, consider the fraction $\frac{1}{4}$. According to the theorem, if we find the representation of $\frac{1}{4}$ in the bases 2, 3, 4, and 5, together with the corresponding keys, then we can easily get the digital representation of $\frac{1}{4}$ in any base. Recall that the key $\langle k_1 \cdots k_\ell \rangle$ associated with $\frac{1}{m}$ in base b is defined by $k_i = (b^{i-1} \pmod{m})$, where ℓ is either the length of the fundamental period of $\frac{1}{m}$ or the length of its nontrivial fractional part. Thus

$$\tfrac{1}{4} = (0.01)_2 \to \langle 12 \rangle, \qquad \tfrac{1}{4} = (0.\overline{02})_3 \to \langle 13 \rangle,$$
$$\tfrac{1}{4} = (0.1)_4 \to \langle 1 \rangle, \qquad \tfrac{1}{4} = (0.\overline{1})_5 \to \langle 1 \rangle.$$

Hence, $\frac{1}{4} = (0.13)_6$ since $01 + \mathit{12} = 13$, and $\frac{1}{4} = 0.25$ because $13 + \mathit{12} = 25$. Similarly, using $\langle 1 \rangle$ as key, one gets for instance

$$\tfrac{1}{4} = (0.2)_8, \quad \tfrac{1}{4} = (0.4)_{16}, \quad \text{and} \quad \tfrac{1}{4} = (0.\overline{3})_{13}.$$

In particular, in base 2009, we have $\frac{1}{4} = (0.\overline{[502]})_{2009}$.

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A Waiting-Time Surprise

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Let $x_1, x_2, x_3, \ldots$ be a sequence of numbers chosen randomly (and uniformly) from the unit interval $0 < x < 1$. For each real number $t \geq 0$, the first n for which

$$x_1 + x_2 + \cdots + x_n > t$$

is a waiting-time random variable; let $E(t)$ denote its expected value. In this note, we express $E(t)$ as a sum of elementary functions of t, and show that it is asymptotic to a linear function.

The data in Table 1 was generated by doing a million trials for each of four target values t. The frequency table shows how often each stopping index n was obtained, and the averages appear at the bottom.

Table 1.

n	$t = 0.3$	$t = 0.5$	$t = 0.8$	$t = 1.0$
1	699876	499770	198960	0
2	255258	375063	480625	499484
3	40345	104297	234716	334490
4	4171	18272	68642	124172
5	330	2358	14346	33533
6	20	222	2344	6987
7	0	18	326	1152
8	0	0	34	161
9	0	0	6	16
10	0	0	1	4
11	0	0	0	1
average	1.349881	1.649123	2.227338	2.718260

Based on this data, the following result appears plausible:

Theorem 1. *For* $0 \le t \le 1$, $E(t) = e^t$.

Proof. For each positive integer n, let $p_n(t)$ denote the probability that

$$x_1 + x_2 + x_3 + \cdots + x_{n-1} \le t < x_1 + x_2 + x_3 + \cdots + x_n.$$

In other words, $p_n(t)$ is the probability that n is the waiting time for target t. It is not difficult to see that $p_1(t) = 1 - t$. In general, the polynomial formula

$$p_n(t) = \frac{1}{(n-1)!}\, t^{n-1} - \frac{1}{n!}\, t^n$$

applies for all $0 \le t \le 1$. Once this formula for $p_n(t)$ is established, it is a routine exercise to show that

$$E(t) = \sum_{n=1}^{\infty} n \cdot p_n(t) = e^t.$$

To establish the formula for $p_n(t)$, notice that $\frac{1}{(n-1)!}\, t^{n-1}$ is the volume of the $(n-1)$-dimensional polytope defined by the inequalities $x_1 \ge 0$, $x_2 \ge 0, \ldots,$ $x_{n-1} \ge 0$, and $x_1 + x_2 + \cdots + x_{n-1} \le t$. It is also the volume of the n-dimensional prism defined by the additional inequality $0 \le x_n \le 1$. This prism includes all positive solutions to the inequality $x_1 + x_2 + \cdots + x_n \le t$, whose probability is $\frac{1}{n!}\,t^n$. Thus

$$\frac{1}{(n-1)!}\,t^{n-1} - \frac{1}{n!}\,t^n = p_n(t)$$

is the probability that

$$x_1 + x_2 + \cdots + x_{n-1} \le t < x_1 + x_2 + \cdots + x_n. \qquad \blacksquare$$

The case $t = 1$ will be familiar to many. It appeared as a Putnam problem in 1958 (see [**1**]), and a discrete version of the problem was analyzed by Shultz [**2**].

It is clear that $E(t)$ cannot be equal to e^t when t is large. Although the geometric approach used for $0 \le t \le 1$ can be modified to cover additional values of t, it is more efficient to assume that $E(t)$ is continuous for $t \ge 0$ and to apply the methods of calculus from now on. Our recursive approach is to observe that, for $t > 1$,

$$E(t) = 1 + \int_{t-1}^{t} E(u)\,du. \tag{1}$$

In other words, $E(t)$ is 1 more than the simple average of all the expected waiting times $E(t - x_1)$ that could result from choosing x_1; we obtain the integral equation by replacing $t - x_1$ by u. When applied to (1), the Fundamental Theorem of Calculus gives

$$E'(t) = E(t) - E(t-1). \tag{2}$$

Notice that our result $E(t) = e^t$ for $0 \le t < 1$ also follows from (2) if we use the obvious values $E(t) = 0$ for $t < 0$ to extend the definition of E.

We now outline an inductive proof that, for $n \ge 1$ and $n - 1 \le t \le n$,

$$E(t) = \sum_{k=0}^{n-1} (-1)^k \,\frac{e^{t-k}}{k!}(t-k)^k. \tag{3}$$

The upper limit of this sum shows that only those terms for which $t - k$ is nonnegative are included. Assume first that $1 < t \le 2$, where we know that $E(t-1) = e^{t-1}$. It is a straightforward application of (2) to show that $\frac{d}{dt}\left(e^{-t}E(t)\right) = -e^{-1}$. From this it follows that $E(t) = e^t - (t-1)e^{t-1}$, because E is continuous at $t = 1$ and $E(1) = e$. Thus (3) holds for $n = 1$. The induction step follows similarly and is left to the reader.

The jump discontinuity at $t = 0$ forces E to be nondifferentiable at $t = 1$ (the two one-sided derivatives are e from the left and $e - 1$ from the right), but formula (3) shows that E is differentiable everywhere else. If $n > 1$, the difference between the two formulas for $E(n)$ is divisible by $(t-n)^2$, forcing the two one-sided formulas for $E'(n)$ to agree.

The recursive process makes use of the continuity of E when t is a positive integer. The values $E(n)$ are interesting:

$$E(1) = 2.71828182\ldots$$
$$E(2) = 4.67077427\ldots$$
$$E(3) = 6.66656564\ldots$$
$$E(4) = 8.66660449\ldots$$
$$E(5) = 10.6666620\ldots$$

Moreover, the emerging pattern is not confined to integer values of t, as the example $E(4.85) = 10.366656\cdots = 2(4.85) + 0.666656\ldots$ shows. The main purpose of this note is to establish the following asymptotic result:

Theorem 2. *The function E defined by equation* (3) *satisfies* $\lim_{t\to\infty}(E(t) - 2t) = \frac{2}{3}$.

Equation (3) does not seem to be of much help in establishing the asymptotic behavior of $E(t)$, but the integral equation (1) and the derived equation (2) do pay dividends. Notice, in particular, that the derivatives of E also satisfy (2). This suggests that we look for an integral equation, similar to (1), that applies to them.

We are thus led to consider functions f that have the *average-value property*, which means that f is continuous for $t \geq 1$ and $f(t) = \int_{t-1}^{t} f(u)\,du$ holds for $t \geq 2$. Unless f is constant, it is clear that f must attain values above and below $f(t)$ on the interval $(t-1, t)$. It is plausible that the continuous averaging process dissipates this variability, forcing $f(t)$ to approach a limit as $t \to \infty$. (Since we expect $f = E'$ to approach a limit, this is exactly what we want to happen.) Furthermore, this limit (if it exists) is determined by the values of f on any unit interval, and it is therefore reasonable to try to express the limit as a weighted average of these values. The recursive nature of f suggests that the weighting function should increase linearly, starting with 0 at the lower limit of the integral. In the trivial case where f is constant, it is easily seen that the formula $\int_{t-1}^{t} 2(u-t+1)f(u)\,du$ produces the correct value. Furthermore, for all functions of interest, the value produced by this integration formula does not depend on the choice of interval:

Lemma 1. *Assume that the continuous function f has the average-value property. Then the function $F(t) = \int_{t-1}^{t}(u-t+1)f(u)\,du$ is constant for $t \geq 2$.*

Proof. As above, the Fundamental Theorem of Calculus yields $f'(t) = f(t) - f(t-1)$ for $t \geq 2$. Notice that $F(t) = \int_{t-1}^{t} uf(u)\,du - (t-1)f(t)$. A short calculation now shows that $F'(t) = 0$. ■

It is shown next that the common value of these integrals *is* the desired limit.

Lemma 2. *If f has the average-value property, then*

$$\lim_{t\to\infty} f(t) = \int_{1}^{2} 2(u-a)f(u)\,du.$$

Proof. Let $L = \int_1^2 2(u-a)f(u)\,du$, and let $g(t) = f(t) - L$. It is routine to verify that g also has the average-value property, and that $\int_1^2 2(u-a)g(u)\,du = 0$, so there is no loss of generality in assuming that $L = 0$. Furthermore, nothing is lost by assuming that $f(t)$ is not constant. In this case, Lemma 1 implies that f has both positive and negative values on every interval of length 1.

We now show that the maxima and minima of f on the intervals $I_n = [n-1, n]$ approach 0 as $n \to \infty$, and from this the lemma follows. Since the two arguments are essentially the same, it suffices to give only one.

Let $M_n = f(a_n)$ be the maximum value of f on I_n. It follows from (2) and the differentiability of f that $f(a_n - 1) = f(a_n)$, and so $M_{n-1} \geq M_n$. The non-increasing sequence $\{M_n\}$ is bounded below by 0, thus it must have a limit $M \geq 0$. Suppose that $M > 0$. For sufficiently large t, $M \leq f(t) < 2M$. From

$$f(t) = \int_{t-1}^{t} f(u)\,du \quad \text{and} \quad \int_{t-1}^{t} (u - t + 1) f(u)\,du = 0,$$

it follows that

$$f(t) = \int_{t-1}^{t} (t-u) f(u)\,du < 2M \int_{t-1}^{t} (t-u)\,du = M,$$

which contradicts the definition of M. ■

Corollary. *Let F be continuous for $t \geq 0$, and let k be constant. If*

$$F(t) = k + \int_{t-1}^{t} F(u)\,du$$

for all $t \geq 1$, then

$$\lim_{t\to\infty} (F(t) - 2kt) = -\frac{4k}{3} + \int_0^1 2uF(u)\,du.$$

Proof. We first apply Lemma 2 to the function $f(t) = F'(t)$, which is easily seen to have the average-value property. A routine integration by parts leads us to

$$\lim_{t\to\infty} F'(t) = \int_1^2 2(u-1)F'(u)\,du = 2k.$$

Another short calculation shows that $g(t) = F(t) - 2kt$ also has the average-value property. It therefore follows from Lemma 2 that

$$\begin{aligned}\lim_{t\to\infty} (F(t) - 2kt) &= \int_1^2 2(u-1)(F(u) - 2ku)\,du \\ &= \int_1^2 2(u-1)(F'(u) + F(u-1) - 2ku)\,du \\ &= -\frac{4k}{3} + \int_0^1 2wF(w)\,dw.\end{aligned}$$

■

Finally, we return to the objective function E expressed in equation (3). Because E satisfies (1), the conclusion of Theorem 2 follows from the corollary and the easily verified calculation

$$-\frac{4}{3} + \int_0^1 2ue^u\,du = \frac{2}{3}.$$

■

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