Commensurable Triangles

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The most famous equation in geometry is $a^2 + b^2 = c^2$, from the Pythagorean theorem. Three positive integers (a, b, c) that satisfy this equation are called a Pythagorean triple, and there are known methods to generate them all. For example, $(n^2 - m^2, 2mn, n^2 + m^2)$ is a Pythagorean triple whenever m and n are relatively prime positive integers, with one odd, one even, and $m < n$. Moreover, every Pythagorean triple whose greatest common divisor is 1 is of this form.

In this article, we consider another family of triples, representing triangles in which one angle is a rational multiple of another. These triples also satisfy defining polynomial equations, and our principal objective is to derive formulas that generate all primitive solutions to those equations.

Given a positive integer k, a triangle in which one angle is k times as large as another is called $(1, k)$-*commensurable*, or simply k-*commensurable*. This terminology is also applicable to triples. A triple is called *integral* or *rational* if its members are all integers or all rational numbers, and an integral triple is called *primitive* if there is no integer greater than 1 that divides all three members.

2-Commensurable triangles

Our first example is the 4–6–5 triangle ΔABC shown in Figure 1. It is 2-commensurable, since $\angle B$ is twice as large as $\angle A$. Notice also that $6^2 = 4(4 + 5)$. In fact, the equation

$$b^2 = a(a + c)$$

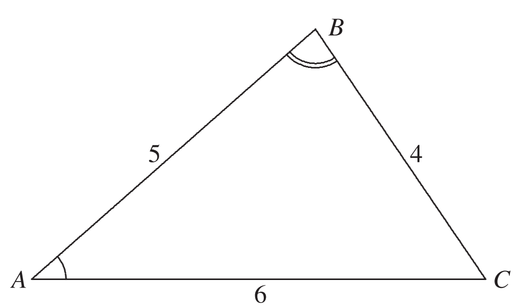

Figure 1.

is a necessary and sufficient condition for $\angle B$ to be twice $\angle A$ in ΔABC. (We are adopting the standard convention that a, b, and c are the lengths of the sides opposite angles A, B, and C.) One way to see this is to notice that the bisector of $\angle B$ splits the triangle into two triangles, one similar to the original, the other isosceles. The reader may enjoy completing this derivation.

A familiar irrational solution to this equation is $(a, b, c) = (1, \sqrt{2}, 1)$, for which $A = 45°$ and $B = 90°$. Another is $(a, b, c) = (1, \sqrt{3}, 2)$, for which $A = 30°$ and $B = 60°$.

Given that $b^2 = a(a + c)$ characterizes 2-commensurable triangles, it is natural to ask for primitive solutions to this equation. Notice that a and $a + c$ must be relatively prime in such a solution. Because every prime divisor of b^2 appears an even number of times, it follows that a must be a perfect square, as must $a + c$.

Any triple of the form $(m^2, mn, n^2 - m^2)$ is a solution to $b^2 = a(a + c)$, and it corresponds to a 2-commensurable triangle if $m < n < 2m$. If m and n are relatively prime, this triple is primitive, and every primitive 2-commensurable triple can be obtained in this way. Table 1 shows the first few examples. The condition $n < 2m$ is needed so that (a, b, c) will satisfy the triangle inequality $c < a + b$. Notice that $\angle A$ is not necessarily the smallest angle in the triangle, nor is $\angle B$ necessarily the largest.

Table 1.

m	n	a	b	c
2	3	4	6	5
3	4	9	12	7
3	5	9	15	16
4	5	16	20	9
4	7	16	28	33
5	6	25	30	11
5	7	25	35	24
5	8	25	40	39
5	9	25	45	56

3-Commensurable triangles

Now consider the 8–10–3 triangle shown in Figure 2. It is 3-commensurable, since $\angle B$ is three times as large as $\angle A$. Notice that $3^2 \cdot 8 = (10 + 8)(10 - 8)^2$. In fact, the equation

$$c^2 a = (b + a)(b - a)^2 \qquad (*)$$

is necessary and sufficient for $\angle B$ to be three times $\angle A$ in ΔABC. One way to derive $(*)$ is draw the trisector of $\angle B$ that lies closer to C. This splits the triangle into two smaller triangles, one similar to ΔABC, the other 2-commensurable. As in the 2-commensurable case, the reader is invited to provide the details.

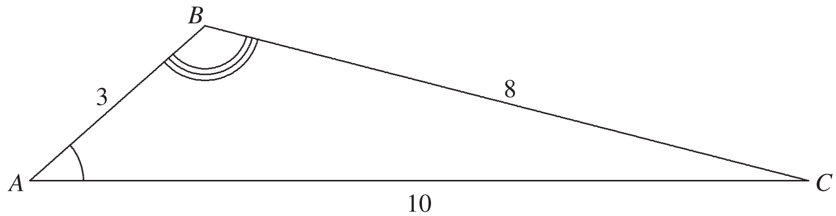

Figure 2.

A familiar irrational solution to $(*)$ is $(a, b, c) = (1, 2, \sqrt{3})$, for which $A = 30°$ and $B = 90°$. Another is $(a, b, c) = (2, 1 + \sqrt{5}, 2)$, for which $A = 36°$ and $B = 108°$.

Let (a, b, c) be a primitive solution to $(*)$, and let m be the greatest common divisor of a and b. Substituting $a = a'm$ and $b = b'm$ into $(*)$ leads to

$$c^2 a' = (b' + a')(b' - a')^2 m^2.$$

Because the triple (a, b, c) is primitive, c is relatively prime to m, and because a' and b' are relatively prime, the same must be true of a' and $(b' + a')(b' - a')^2$. It follows that

$$a' = m^2 \quad \text{and} \quad c^2 = (b' + m^2)(b' - m^2)^2.$$

This in turn implies that $b' + m^2$ must be a perfect square n^2. Thus a primitive 3-commensurable triple must take the form $(a, b, c) = \big(m^3, mn^2 - m^3, n^3 - 2m^2 n\big)$, where m and n are relatively prime. In order that $c > 0$ and the triangle inequality be satisfied, n must lie between $\sqrt{2}m$ and $2m$.

Table 2 shows the first few primitive 3-commensurable triples. As in Table 1, $\angle A$ need not be the smallest angle in the triangle, nor $\angle B$ the largest. Notice, however, that the pairs (m, n) in Table 1 for which $\angle A$ *is* the smallest angle also appear in Table 2. This persistence is a consequence of a recursive approach, which we describe below.

Table 2.

m	n	a	b	c
2	3	8	10	3
3	5	27	48	35
4	7	64	132	119
5	8	125	195	112
5	9	125	280	279
6	11	216	510	539
7	10	343	357	20
7	11	343	504	253
7	12	343	665	552
7	13	343	840	923

k-Commensurable triangles

Figure 3 shows a k-commensurable triangle ΔABC, in which $\overline{BD}$ is drawn so that $\angle DBC$ is congruent to $\angle A$. Triangles BDC and ABC are similar, with $BD/AB = a/b$. Thus $BD = ac/b$ and $DC = a^2/b$. Triangle ABD is $(k-1)$-commensurable. In particular, ΔABD is isosceles when $k = 2$. Hence the equation

$$b^2 - a^2 = ac$$

characterizes 2-commensurable triangles. This equation can in turn be applied recursively to the case $k = 3$, yielding

$$\left(b - \frac{a^2}{b}\right)^2 - \left(\frac{ac}{b}\right)^2 = c\left(\frac{ac}{b}\right).$$

Rearranging this equation and removing an extraneous factor $a + b$ leads to the promised 3-commensurable equation (∗).

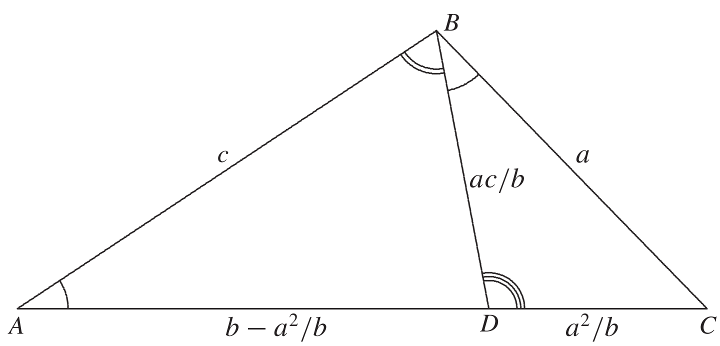

Figure 3.

Although any k-commensurable triangle can be derived from a $(k-1)$-commensurable triangle, not every $(k-1)$-commensurable ΔABC can be enlarged to a k-commensurable triangle. Such an enlargement of $\angle B$ is possible if, and only if, $\angle A$ is smaller than $\angle C$. As Figure 4 shows, this condition is needed so that the new ray through B will intersect $\overrightarrow{AC}$ at D.

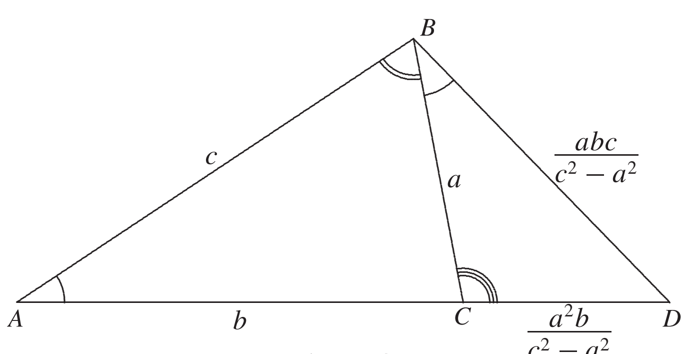

Figure 4.

Notice also that the labels on $\overline{CD}$ and $\overline{BD}$ in Figure 4 make sense only when $a < c$. The reader can obtain them by solving the proportion

$$\frac{CD}{BD} = \frac{BD}{b + CD} = \frac{a}{c},$$

derived from the similar triangles ΔBCD and ΔABD.

Recursive calculations

The preceding discussion shows that every rational k-commensurable triangle can be derived by enlarging a rational $(k-1)$-commensurable triangle. Applied to primitive $(k-1)$-commensurable triples (a, b, c) in which $a < c$, the formula

$$\left(\frac{abc}{c^2 - a^2}, \frac{bc^2}{c^2 - a^2}, c\right)$$

thus produces a rational multiple of every primitive k-commensurable triple. It follows that every primitive k-commensurable triple can be obtained by applying the following formula, removing common factors when necessary.

Recursion 0. If (a, b, c) is an integral $(k-1)$-commensurable triple, and if $a < c$, then $(ab, bc, c^2 - a^2)$ is an integral k-commensurable triple.

Let m and n be relatively prime. Then primitive 1-commensurable (isosceles) triples are described by $(a, b, c) = (m, m, n)$ for $n < 2m$, and primitive 2-commensurable triples are described by $(a, b, c) = (m^2, mn, n^2 - m^2)$ for $m < n < 2m$. When Recursion 0 is applied to $(a, b, c) = (m^2, mn, n^2 - m^2)$, however, the result is

$$(a, b, c) = \left(m^3 n, mn^3 - m^3 n, n^4 - 2m^2 n^2\right),$$

which is n times the earlier formula for primitive 3-commensurable triples.

The reader may wish to use this approach to verify some of the entries in Table 3. Notice that the kth row is a triple (a_k, b_k, c_k) of kth-degree polynomials in m and n.

Table 3.

a_k	b_k	c_k
m	m	n
m^2	mn	$n^2 - m^2$
m^3	$mn^2 - m^3$	$n^3 - 2m^2 n$
m^4	$mn^3 - 2m^3 n$	$n^4 - 3m^2 n^2 + m^4$
m^5	$mn^4 - 3m^3 n^2 + m^5$	$n^5 - 4m^2 n^3 + 3m^4 n$
m^6	$mn^5 - 4m^3 n^3 + 3m^5 n$	$n^6 - 5m^2 n^4 + 6m^4 n^2 - m^6$

Computing the first few rows of Table 3 led us to conjecture that every removed common polynomial factor would itself be found in the table. Given the initial data

$c_0 = 1$ and $(a_1, b_1, c_1) = (m, m, n)$, the rows of Table 3 appear to obey the following recursion:

Recursion 1. For $k > 1$, let $a_k = \dfrac{a_{k-1}b_{k-1}}{c_{k-2}}$, $b_k = \dfrac{b_{k-1}c_{k-1}}{c_{k-2}}$, and $c_k = \dfrac{c_{k-1}^2 - a_{k-1}^2}{c_{k-2}}$.

Working exclusively with this recursion proved to be awkward, however. A closer look at the table prompted an even simpler description:

Recursion 2. For $k > 1$, let $a_k = ma_{k-1}$, $b_k = mc_{k-1}$, and $c_k = nc_{k-1} - m^2c_{k-2}$.

The familiar look of the coefficients in this table eventually inspired this explicit formulation:

Recursion Lemma. *Recursions 1 and 2 produce the same polynomials, namely*

$$a_k = m^k,$$
$$c_k = \sum_{i=0}^{\lfloor k/2 \rfloor} (-1)^i \binom{k-i}{i} m^{2i} n^{k-2i},$$

and

$$b_k = mc_{k-1}.$$

Proof. An inductive demonstration is anchored for small k by the Table 3 data. The following four exercises complete the argument. ∎

Exercise 1. By combining two successive applications of Recursion 2, eliminate n and obtain $c_{k-2}c_k + m^2c_{k-2}^2 = c_{k-1}^2 + m^2c_{k-3}c_{k-1}$. This is valid for $k \geq 3$.

Exercise 2. Obtain $c_{k-2}c_k = c_{k-1}^2 - m^{2k-2}$, the critical part of Recursion 1, by creating a telescoping sum from the result of Exercise 1. This is valid for $k \geq 2$.

Exercise 3. Assume for the purposes of induction that the triples (a_i, b_i, c_i) generated by the two recursions agree for all $i < k$. Combine Exercise 2 and Recursion 1 to show that the two recursions then agree on (a_k, b_k, c_k).

Exercise 4. *Apply $c_k = nc_{k-1} - m^2c_{k-2}$ (from Recursion 2) to establish the explicit formula for c_k, assuming its validity for c_{k-1} and c_{k-2}.*

The first theorem

Now consider a k-commensurable triangle obtained by evaluating the polynomials a_k, b_k, and c_k at a suitable pair (m, n) of relatively prime positive integers. The explicit formulas make it clear that the integers $a_k(m, n)$ and $c_k(m, n)$ are relatively prime. Hence the triple $(a_k(m, n), b_k(m, n), c_k(m, n))$ is primitive. The meaning of "suitable" is clarified next.

Cosine Lemma. *If ΔABC is a rational k-commensurable triangle, then $(2 \cos A)m = n$ for some pair of relatively prime positive integers m and n.*

Proof. This is obvious when $k = 1$, and the reader might enjoy using the law of cosines to verify the formula in the case $k = 2$. However, because every k-commensurable triangle can be obtained by enlarging a $(k-1)$-commensurable triangle that shares $\angle A$, no additional work is actually needed. ■

This accomplishes one of the major objectives of this article:

Theorem 1. *For any positive integer k, the polynomials*

$$a_k = m^k$$
$$b_k = mc_{k-1}$$
$$c_k = \sum_{i=0}^{\lfloor k/2 \rfloor} (-1)^i \binom{k-i}{i} m^{2i} n^{k-2i}$$

generate all primitive k-commensurable triples, when they are evaluated at pairs (m, n) *of relatively prime positive integers that satisfy* $\left(\cos \frac{180^\circ}{k+1}\right) m < n < 2m$.

Proof. The sum of the angles of any triangle is 180°, and a k-commensurable triangle ΔABC contains $k+1$ copies of $\angle A$. Thus $(k+1)A < 180^\circ$ and $\cos \frac{180^\circ}{k+1} < \cos A < 1$. The result now follows from the Cosine Lemma. ■

The commensurable triangle theorem

If k/h is a rational number in lowest terms, then a triangle in which one angle is k/h times as large as another is called (h, k)-*commensurable*. The preceding discussion has dealt thoroughly with the case $h = 1$. Our goal is to use Table 3 to build a complete catalogue of primitive (h, k)-commensurable triples. The next example illustrates the case $h = 2$ of an impending theorem.

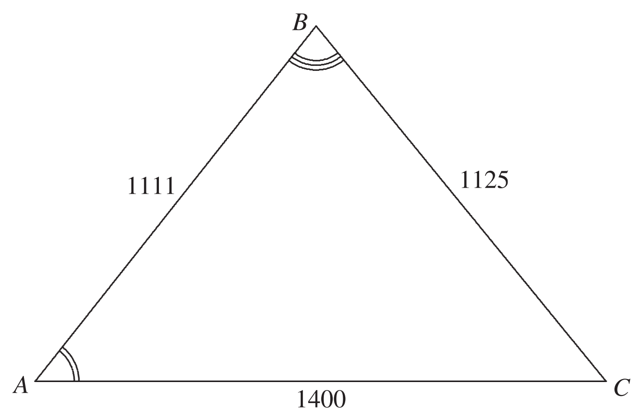

Figure 5.

The 1125–1400–1111 triangle shown in Figure 5 is (2, 3)-commensurable, since $\angle B$ is $1\frac{1}{2}$ times as large as $\angle A$. The reader can also verify that

$$(1400^2 - 1125^2)^2 + 1400 \cdot 1111(1400^2 - 1125^2) = 1125^2 \cdot 1111^2.$$

In fact, the equation

$$\left(b^2 - a^2\right)^2 + bc(b^2 - a^2) = a^2c^2$$

is necessary and sufficient for $\angle B$ to be $1\frac{1}{2}$ times $\angle A$ in ΔABC.

One way to see this is to notice that the trisector of $\angle B$ that is closer to A splits the triangle into two triangles, one similar to the original, the other 2-commensurable. This approach can also be used to show that primitive solutions to this equation have the form $(m^3n, m^2n^2 - m^4, n^4 - 3m^2n^2 + m^4)$, where m and n are relatively prime and $(2\cos 36^\circ)m < n < 2m$. Table 4 displays a few examples.

Table 4.

m	n	a	b	c
3	5	135	144	31
4	7	448	528	305
5	9	1125	1400	1111
6	11	2376	3060	2869
7	12	4116	4655	1969
7	13	4459	5880	6119
8	13	6656	6720	209
8	15	7680	10304	11521
9	16	11664	14175	9889

The formula for primitive (2, 3)-commensurable triples has the form

$$(c_1(m,n)a_3(m,n), a_1(m,n)b_3(m,n), c_4(m,n)),$$

which is a special case of the following definition. For any relatively prime pair (h, k) of positive integers, let

$$a_{h,k}(m,n) = c_{h-1}(m,n)a_k(m,n),$$
$$b_{h,k}(m,n) = a_{h-1}(m,n)b_k(m,n), \quad \text{and}$$
$$c_{h,k}(m,n) = c_{k+h-1}(m,n).$$

(Notice that $(a_{1,k}, b_{1,k}, c_{1,k}) = (a_k, b_k, c_k)$.)

Theorem 2. *Given a pair (m, n) of relatively prime positive integers that satisfy the constraint $\left(2\cos\frac{180^\circ}{k+h}\right)m < n < 2m$, the triple $(a_{h,k}(m,n), b_{h,k}(m,n), c_{h,k}(m,n))$ is primitive and (h, k)-commensurable. Conversely, any primitive (h, k)-commensurable triple can be obtained in this way.*

This is the principal result of this article. To avoid repetitious detail, the proof is outlined as a sequence of lemmas and exercises. Our familiar dissection tactic is used recursively.

Any rational k-commensurable triangle $\triangle ABC$ has the property that $(2\cos A)m = n$ for some relatively prime positive integers m and n. These are still the only angles that can be used to build a rational (h, k)-commensurable triangle $\triangle ABC$.

Rational Lemma. *Let h and k be relatively prime positive integers, and $\triangle ABC$ be a rational (h, k)-commensurable triangle. Then there are relatively prime positive integers m and n such that $A = h\theta$, $B = k\theta$, and $(2\cos\theta)m = n$.*

Exercise 5. Prove this lemma, assuming that $h < k$. Start by locating D on $\overline{AC}$ so that $\angle DBC$ is congruent to $\angle BAC$. Explain why $\triangle ABD$ is $(h, k - h)$-commensurable and rational. To finish the proof, it will be necessary to know that h and k are relatively prime.

The proof of Theorem 2 now proceeds by dissecting $\triangle ABC$ in a different fashion. Locate E on $\overline{BC}$ so that $\angle CAE = \theta$. Notice that $\triangle AEC$ is $(k + h - 1)$-commensurable and $\triangle ABE$ is $(h - 1, k)$-commensurable. Because the shape of a triangle is determined once two of its angles are known, it follows that $\triangle AEC$ is rational, and that the lengths of its sides are proportional to $a_{k+h-1}(m, n)$, $b_{k+h-1}(m, n)$, and $c_{k+h-1}(m, n)$. In the same fashion, it follows by induction that the sides of $\triangle ABE$ are proportional to $a_{h-1,k}(m, n)$, $b_{h-1,k}(m, n)$, and $c_{h-1,k}(m, n)$.

Exercise 6. The sides of a rational (h, k)-commensurable $\triangle ABC$ are proportional to

$$a_{k+h-1}(m, n)b_{h-1,k}(m, n) + c_{k+h-1}(m, n)a_{h-1,k}(m, n),$$
$$b_{k+h-1}(m, n)b_{h-1,k}(m, n),$$
$$c_{k+h-1}(m, n)c_{h-1,k}(m, n),$$

for some relatively prime positive integers m and n. Show this by using a consistent labeling of the sides of $\triangle AEC$ and $\triangle ABE$.

A technical lemma is now needed to establish that the integer $c_{k+h-2}(m, n)$ is a common factor of the preceding triple of integers.

Lemma. *For integers $h > 1$ and $k \geq 1$,*

$$m^{2h-3}b_k = c_{h-1}c_{k+h-2} - c_{h-2}c_{k+h-1}.$$

Exercise 7. Prove the lemma by induction on h. Notice that the base case $h = 2$ is a direct consequence of Recursion 2.

Exercise 8. Use the lemma and induction to prove the polynomial identities

$$a_{k+h-1}b_{h-1,k} + c_{k+h-1}a_{h-1,k} = c_{k+h-2}a_{h,k},$$
$$b_{k+h-1}b_{h-1,k} = c_{k+h-2}b_{h,k}, \quad \text{and}$$
$$c_{k+h-1}c_{h-1,k} = c_{k+h-2}c_{h,k}.$$

Conclude that the sides of any integral (h, k)-commensurable triangle are proportional to the integers $a_{h,k}(m, n)$, $b_{h,k}(m, n)$, and $c_{h,k}(m, n)$ for suitable relatively prime integers m and n.

Exercise 9. *Assuming that* (h, k) *and* (m, n) *are pairs of relatively prime integers, show that the triple* $(a_{h,k}(m, n), b_{h,k}(m, n), c_{h,k}(m, n))$ *is primitive.*

This concludes the proof of Theorem 2.

Wrap-up, additional examples, and questions

One obvious question remains: Can a non-equilateral triangle have integral sides and three commensurable angles? The answer is *no*. For a short proof, see [**1**, p. 228].

Among the primitive 2-commensurable triples are those for which n has the extreme value $m + 1$. For any triple $(m^2, m(m + 1), 2m + 1)$, the law of cosines shows that

$$\cos A = \frac{m+1}{2m} \quad \text{and} \quad \cos B = \frac{-m^2 + 2m + 1}{2m^2}.$$

Thus these triangles approach the degenerate triangle whose angles are $A = 60°$, $B = 120°$, and $C = 0°$.

At the other extreme, $n = 2m - 1$ produces triples $(m^2, 2m^2 - m, 3m^2 - 4m - 1)$. The corresponding triangles approach the shape of the degenerate triangle $(1, 2, 3)$, whose angles are $A = 0°$, $B = 0°$, and $C = 180°$.

The sequence of primitive 2-commensurable examples

$$(4, 6, 5),\ (25, 35, 24),\ (144, 204, 145),\ (841, 1189, 840),$$
$$(4900, 6930, 4901),\ \ldots$$

approaches the shape of an isosceles right triangle. The corresponding pairs (m, n) are $(2, 3), (5, 7), (12, 17), (29, 41), \ldots,$ where (m, n) is followed by $(m + n, 2m + n)$. The ratios $\frac{n}{2m}$ approach $\frac{1}{2}\sqrt{2}$.

The sequence of primitive 3-commensurable examples

$$(8, 10, 3),\ (343, 665, 552),\ (17576, 35074, 30285),$$
$$(912673, 1825055, 1580208),\ \ldots$$

approaches the shape of a 30°–60° right triangle. The corresponding pairs (m, n) are $(2, 3), (7, 12), (26, 45), (97, 168), \ldots,$ where (m, n) is followed by $(2m + n, 3m + 2n)$. The ratios $\frac{n}{2m}$ approach $\frac{1}{2}\sqrt{3}$.

For each k, there are infinitely many relatively prime pairs (m, n) for which (a_k, b_k, c_k) is a k-commensurable triangle, since there are infinitely many rational numbers between $\cos \frac{180°}{k+1}$ and 1. This interval becomes vanishingly small as k increases, forcing the table of k-commensurable examples to begin with large values of m. Methods of calculus show that $m > \left(\frac{k+1}{\pi}\right)^2$, so the smallest side of a k-commensurable triangle exceeds $\left(\frac{k+1}{\pi}\right)^{2k}$. For example, $(43046721, 58429017, 16657264)$ is the smallest 8-commensurable triangle, produced by $m = 9$ and $n = 17$.

We conclude with three questions. We would like to know an answer for the third.

Question 1. Can the shape of every irrational commensurable triangle be approximated to arbitrary precision by a rational commensurable triangle?

Question 2. Given a triangle ΔABC, the lengths of whose sides form a primitive commensurable triple (a, b, c), is it possible that the area of ΔABC is rational?

Question 3. Given a pair (h, k) of relatively prime positive integers, it is now clear that there is a polynomial equation $P_{h,k}(a, b, c) = 0$ with integer coefficients that defines (h, k)-commensurable triangles ΔABC. For example, $P_{2,3}(a, b, c) = (b^2 - a^2)^2 + bc(b^2 - a^2) - a^2c^2$. This article provides an explicit description of all rational solutions to such equations. Is there an explicit formula for the polynomials $P_{h,k}(a, b, c)$ themselves?

References

1. J. H. Conway and R. K. Guy, *The Book of Numbers*, Springer-Verlag, 1996.